Universidad Politécnica de Madrid (UPM)
Escuela Técnica Superior de Ingenieros Aeronáuticos
Subdirección de Investigación, Doctorado y Posgrado
Plaza Cardenal Cisneros, 3
Madrid 28040, España

Partial Differential Equations (PDE)

Prof. Vega, José Manuel
Prof. Madruga, Santiago
Assignment #2

Fasolino, José Daniel	Y2888590-B / AA3671076
Quiles Ibernón, Pablo	23047308-C
Salguero Aller, Javier	71455464-F
Sanchez Carmona, Alejandro	49096009-W

Consider the two-dimensional wave equation:

$$\partial_{tt}u - \Delta u = f(x,y,t) \;\; if \; t > 0, \quad u(x,y,0) = \partial_t u(x,y,o) = 0$$

And heat equation

$$\partial_t u - \Delta u = f(x,y,t) \; if \;\; t > 0; \quad u(x,y,0)$$

Where the forcing function f is given by

$$f(x,y,t) = cos2t \exp[-(2x^2 + 3y^2)/4]$$

1. Solve the wave equation in the whole plane, with $u \to 0$ as $x^2 + y^2 \to \infty$, using the Green's function.

2. Solve the wave equation in the domain Ω: $-1 < x < 1, -1 < y < 1,$ with $u = 0$ at $\partial\Omega$ using a spectral representation.

3. For both the unbounded and bounded domain:

   3.1 Elucidate whether the solutions are in phase with the forcing.
   3.2 Compare the CPU time that is required to construct a snapshot of the solution calculated in questions 1 and 2 in the domain Ω in a 100 x 100 equispaced grid at $t = \pi/2$.
   3.3 Construct the appropriate graphical representations of the solution calculated in questions 1 and 2 to illustrate the solution in the domain Ω as time proceeds.

4. Repeat questions 2, 3, 3.1 and 3.3 for the damped wave equation

$$\partial_{tt}u + \varepsilon\partial_t u - \Delta u = f(x,y,t) \;\; if \; t > 0, \quad u(x,y,0) = \partial_t u(x,y,0) = 0$$

with $\varepsilon = 0.01$

1. Mathematical Analysis

1.1 Wave equation, using Green's function

The 2D wave equation with a forcing is given by:

$$\partial_{tt}u - \Delta u = f(x,y,t)$$

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \frac{\partial^2 g}{\partial t^2} - \delta(x-\xi)\delta(y-\eta)\delta(t-\tau) \qquad [1\text{- }1]$$

For simplification, from this point $x-\xi$ will be called x; $y-\eta$ will be called y and $t-\tau$ will be called t. The condition at an initial time,

$$g|_{t=0} = \frac{\partial g}{\partial t}|_{t=0} = 0$$

And the convergence condition at infinity:

$$g|_{\sqrt{x^2+y^2}\to\infty} = \frac{\partial g}{\partial x}|_{\sqrt{x^2+y^2}\to\infty} = \frac{\partial g}{\partial y}|_{\sqrt{x^2+y^2}\to\infty} = 0$$

Whit two space variable and one time variable, a triple integral transform is applied to the differential equation [1- **1**]. The Laplace transform with respect to the time variable:

$$g^*(s) = \int_0^\infty g(t)e^{-st}dt$$

And the double Fourier transforms with respect to the space variables:

$$\bar{g}(\xi) = \int_{-\infty}^{\infty} g(x)e^{+i\xi x}dx\,; \quad g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{g}(\xi)e^{-i\xi x}d\xi$$

$$\tilde{g}(\eta) = \int_{-\infty}^{\infty} g(y)e^{+i\eta x}dy\,; \quad g(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{g}(\eta)e^{-i\eta x}d\eta$$

Applying this triple integral transform with the initial and boundary conditions, the wave equation is transformed to the simple algebraic equation for the unknown function $\tilde{\bar{g}}^*$:

$$-\xi^2\tilde{\bar{g}}^* - \eta^2\tilde{\bar{g}}^* = s^2\tilde{\bar{g}}^* - 1$$

$$\tilde{\bar{g}}^* = \frac{1}{\xi^2+\eta^2+s^2} \qquad [1\text{- }2]$$

For the inversion, three inversion integrals must be carried out successively. The first is the Fourier inversion integral with respect to η. Then the system is reduced to a semi-infinite integral:

$$\bar{g}^* = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\xi^2+\eta^2+s^2}e^{-i\eta y}d\eta \quad \rightarrow \quad \bar{g}^* = \frac{1}{\pi}\int_{0}^{\infty}\frac{\cos(\eta y)}{\eta^2+\sqrt{\xi^2+s^2}}d\eta$$

This can be evaluated using:

$$\int_0^{\infty}\frac{1}{x^2+a^2}\cos(xy)\,dx = \frac{\pi}{2a}\exp(-a|y|)$$

Then the first Fourier inversion integral yields:

$$\bar{g}^* = \frac{1}{2\sqrt{\xi^2+s^2}}\exp\left\{-|y|\sqrt{\xi^2+s^2}\right\}$$

Secondly, the inversion integral with respect to ξ:

$$g^* = \frac{1}{2\pi}\int_0^{\infty}\frac{1}{\sqrt{\xi^2+s^2}}\exp\left\{-|y|\sqrt{\xi^2+s^2}\right\}e^{-i\xi x}d\xi$$

Where the next integration formula can be applied:

$$\int_0^{\infty}\frac{1}{\sqrt{x^2+a^2}}\exp\left(-\sqrt{x^2+a^2}\right)\cos(bx)\,dx = K_0\left(a\sqrt{b^2+c^2}\right)$$

Where K_0, is the zeroth order modified Bessel function of the second kind. Applying this to g^*

$$g^* = \frac{1}{2\pi}K_0\left(s\sqrt{x^2+y^2}\right)$$

For last, the Fourier inversion is made:

$$G = \frac{1}{2\pi}L^{-1}\left[K_0\left(s\sqrt{x^2+y^2}\right)\right]$$

For which the following inversion formula can be applied:

$$L^{-1}[K_0(as)] = \frac{H(t-a)}{\sqrt{t^2-a^2}} \quad \rightarrow \quad \begin{Bmatrix} 0 & t<a \\ \dfrac{1}{\sqrt{t^2-a^2}} & t>a \end{Bmatrix}$$

Where H, is the Heaviside's unit step function. Applying this inversion formula to the equation:

$$G = \frac{1}{2\pi}\frac{H(t-r)}{\sqrt{t^2-r^2}} = \frac{1}{2\pi}\begin{Bmatrix} 0 & t<r \\ \sqrt{t^2-r^2} & t>r \end{Bmatrix}$$

Where $r = \sqrt{x^2+y^2}$, consequently the expression for Green's function of this equation is expressed as:

$$G = \frac{1}{2\pi}\frac{H(t-r)}{\sqrt{t^2-r^2}}$$

Taking back the change of variable used at the beginning of this section, the Green function is represented as:

$$G = \frac{1}{2\pi}\frac{H\left((t-\tau)-\sqrt{(x^2-\xi^2)+(y^2-\eta^2)}\right)}{\sqrt{(t-\tau)^2-(x^2-\xi^2)+(y^2-\eta^2)}}$$

Thus, the solution of the wave equation in the whole plane:

$$u = \frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{H\left((t-\tau)-\sqrt{(x^2-\xi^2)+(y^2-\eta^2)}\right)}{\sqrt{(t-\tau)^2-\left((x^2-\xi^2)+(y^2-\eta^2)\right)}}\right].f(\xi,\eta,\tau)d\xi\, d\eta\, d\tau \qquad [1\text{-}3]$$

Calling $r = \sqrt{(x^2-\xi^2)+(y^2-\eta^2)}$, and fixing values for x and y, then the solution is non-zero when $t-\tau > r$. Meaning that the wave caused by the forcing term applied at this point (x,y) takes a time t to reach another point located at a distance $r = t$.

1.2 Wave equation in a bounded domain

To reduce the calculus needed to answer all the questions, question 4 is going to be approached at first, and for section 2 and 3 the damping will be taken as 0 when needed.

$$\partial_{tt}u + \varepsilon\partial_t u - \Delta u = f(x,y,t) \;\; if\ t>0, \quad u(x,y,0) = \partial_t u(x,y,0) = 0 \qquad [1\text{-}4]$$

To solve the Duhamel's principle will be applied. The Duhamel's principle is a general method for obtaining solutions to inhomogeneous equations by reducing using the method of variation of parameters:

$$\partial_{tt}v + \varepsilon\partial_t v - \Delta v = 0$$
$$v(x,y,t-\tau,\tau)|_{t=0} = 0 \;\; at\ \partial\Omega \;\wedge\; \partial_t v(x,y,t-\tau,\tau) = f(x,y,\tau)$$

Where $s = t - \tau$ and $u \rightarrow v$. Using separation of variables:

$$v = X(x)Y(y)S(s)T(\tau) \qquad [1\text{- }5]$$

The system in equation [1- **4**] becomes:

$$S''X\,Y\,T + \varepsilon S'X\,Y\,T - X''TYS - Y''XTS = 0$$

Dividing by $TXYS$:

$$\frac{S''}{S} + \varepsilon\frac{S'}{S} = \frac{X''}{X} + \frac{Y''}{Y}$$

Knowing that when you have a differential equation dependent of a variable equal to another differential equation dependent on a different variable (or variables), these can only be equal if they are equal to a constant. In this case the constant will be called $-\lambda^2$

$$\frac{S''}{S} + \varepsilon\frac{S'}{S} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2 \qquad [1\text{- }6]$$

Solving for S:

$$\frac{S''}{S} + \varepsilon\frac{S'}{S} = -\lambda^2$$

$$S'' + \varepsilon S' + \lambda^2 S = 0$$

This can be seen as $s^2 + \varepsilon s + \lambda^2$, and solved finding the roots

$$s = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 - 4\lambda^2}}{2} \;\rightarrow\; s = -\frac{\varepsilon}{2} \pm \sqrt{\left(\frac{\varepsilon}{2}\right)^2 - \lambda^2}$$

Knowing that ε is small, then the solutions will be complex. For simplicity at the moment of writing the system; $i\sqrt{\lambda^2 - (\varepsilon/2)^2}$, will be called w. The solution of the ODE

$$S(s) = e^{(-\varepsilon/2)s}[A sin(ws) + B cos(ws)]$$

Which because of the initial conditions:

$$S(s) = e^{(-\varepsilon/2)s} A sin(ws) \qquad [1\text{- }7]$$

Then the right hand side of the equation [1- **6**] is solved:

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2 \rightarrow \frac{X''}{X} = -\lambda^2 - \frac{Y''}{Y}$$

As before, both sides will be equal to a constant (in this case called $-\mu^2$):

$$\frac{X''}{X} = -\lambda^2 - \frac{Y''}{Y} = -\mu^2 \qquad [1\text{- }8]$$

Solving in X:

$$\frac{X''}{X} = -\mu^2 \rightarrow \quad X'' + \mu^2 X = 0$$
$$X = Csin(\mu x) + Dcos(\mu x)$$

As the domain is bounded to $\Omega: -1 < x < 1$:

$$x = 1: \ \ 0 = Csin\mu + Dcos\mu$$
$$x = -1: \ \ 0 = -Csin\mu + Dcos\mu$$
$$If\ C = 0: \ 0 = Dcos\mu \ \rightarrow \ \ cos\mu = 0 \rightarrow \ \mu = \frac{m\pi}{2}; m: odd\ number$$
$$If\ D = 0: \ 0 = Csin\mu \ \rightarrow \ sin\mu = 0 \ \ \rightarrow \ \ \mu = m\pi; \ m: \ \in \mathbb{N}$$

$$X(x) = D_m cos\left(\frac{m\pi x}{2}\right) \ \ \text{or/and} \ \ X(x) = C_m \sin(m\pi x) \qquad [1\text{- }9]$$

Solving in Y:

$$\frac{Y''}{Y} + \lambda^2 = \mu^2; \ \ \lambda^2 - \mu^2 = p^2 \rightarrow \frac{Y''}{Y} = -p^2 \rightarrow \ Y'' + p^2 Y = 0$$
$$Y = Esin(py) + Fcos(py)$$

Using again the boundary condition $\Omega: -1 < y < 1$ where the system is bounded:

$$y = 1: \ \ 0 = Esinp + Fcosp$$
$$y = -1: \ \ 0 = -Esinp + Fcosp$$
$$If\ E = 0: \ 0 = Fcosp \ \rightarrow \ \ cosp = 0 \rightarrow \ p = \frac{n\pi}{2}; n: odd\ number$$
$$If\ F = 0: \ 0 = Esinp \ \rightarrow \ sinp = 0 \ \ \rightarrow \ \ p = n\pi; \ n: \ \in \mathbb{N}$$

$$Y = F_n cos\left(\frac{n\pi y}{2}\right) \ \ \text{or/and} \ \ Y = E_n \sin(n\pi y) \qquad [1\text{- }10]$$

Now, as $\lambda^2 + \mu^2 = p^2$; then there will be 4 possible combinations for λ:

$$\lambda^2 = p^2 + \mu^2$$
$$\lambda^2 = \frac{n\pi}{2} + \frac{m\pi}{2} \ \rightarrow \lambda = \sqrt{\frac{n\pi}{2} + \frac{m\pi}{2}}$$
$$\lambda^2 = \frac{n\pi}{2} + m\pi \ \rightarrow \lambda = \sqrt{\frac{n\pi}{2} + m\pi}$$

$$\lambda^2 = n\pi + \frac{m\pi}{2} \rightarrow \lambda = \sqrt{n\pi + \frac{m\pi}{2}}$$
$$\lambda^2 = n\pi + m\pi \rightarrow \lambda = \sqrt{n\pi + m\pi}$$

Substituting equations [1- **7**],[1- **9**],[1- **10**], into [1- **5**], the system v can be expressed as follows:

$$v = Ae^{(-\varepsilon/2)s} sin(ws) \, [Csin(\mu x) + Dcos(\mu x)] \, [Esin(py) + Fcos(py) \,] T(\tau)$$

At this point, in order to narrow down the possibilities; the 4 combinations need to be analyzed. The most important think to be taken in to account is the symmetry related to the cosine and sine functions; as in this system either sine or cosine relations can be taken for the X and Y ode's, the cosine relations are chosen due to its even symmetry with the 0 because the system also has symmetry with respect to 0 at the boundary conditions; thus the system can be represented as:

$$\sum_n \sum_m A_{nm} \, sin(ws) \, e^{(-\varepsilon/2)s} \left[cos\left(\frac{\left(\frac{2m+1}{2}\right)\pi x}{2} \right) \right] \left[cos\left(\frac{\left(\frac{2n+1}{2}\right)\pi y}{2} \right) \right] T(\tau)$$

Applying the Duhamel boundary condition $(\partial_t v(x, y, t-\tau, \tau) = f(x, y, \tau))$ at $s = 0$. Taking into account that the derivative with respect to t is equivalent to the derivative with respect to s:

$$A_{nm} \, w \, cos(ws) \, e^{\left(-\frac{\varepsilon}{2}\right)s} \left[cos\left(\frac{\left(\frac{2m+1}{2}\right)\pi x}{2} \right) \right] \left[cos\left(\frac{\left(\frac{2n+1}{2}\right)\pi y}{2} \right) \right] T(\tau) = f(x, y, \tau)$$

Using separation of variables for $f(x, y, \tau) \rightarrow f_x(x) f_y(y) f_\tau(\tau)$

$$wT(\tau) = f_\tau(\tau) \;\rightarrow\; T(\tau) = \frac{f_\tau(\tau)}{w}$$

$$A_{nm} = \frac{< f_x \, , \cos\left(\frac{\left(\frac{2m+1}{2}\right)\pi x}{2} \right) >< f_{y,} \cos\left(\frac{\left(\frac{2n+1}{2}\right)\pi y}{2} \right) >}{< \cos^2\left(\frac{\left(\frac{2m+1}{2}\right)\pi x}{2} \right) >< \cos^2\left(\frac{\left(\frac{2n+1}{2}\right)\pi y}{2} \right) >}$$

Where f_x will be associated to the $\exp(-x^2/2)$ and the f_y to the $\exp(-3y^2/4)$.

2 Numerical evaluations

2.1 Solution in phase with the forcing term

In order to elucidate if the solution obtained through spectral representation is in phase or not with the forcing term the system is analyzed at some fixed point. In all the representation the red line is the forcing and the blue line is system reaction to this forcing:

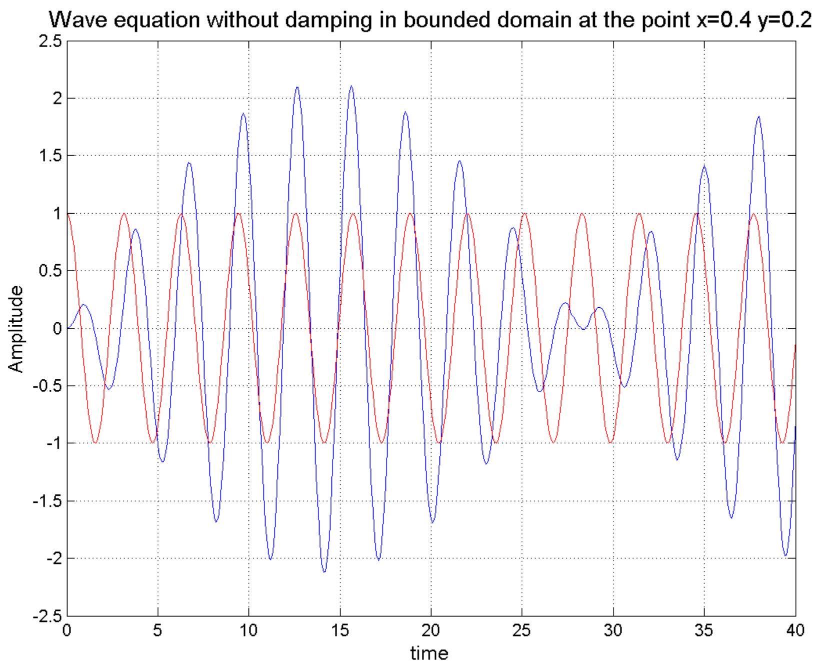

Figure 2-1 Phase without damping

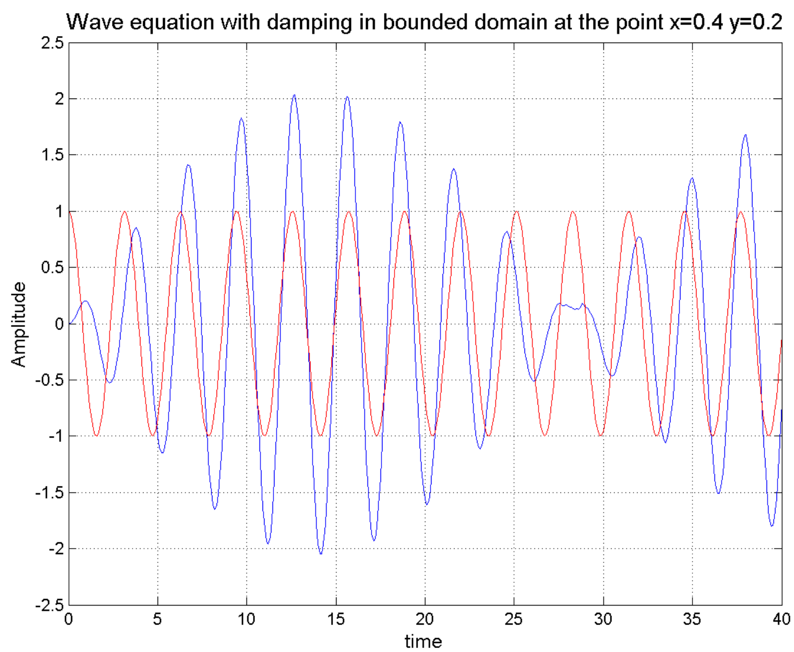

Figure 2-2 Phase with damping

As can be seen in both, **¡Error! No se encuentra el origen de la referencia.** and **¡Error! No se encuentra el origen de la referencia.** the phase between the forcing term and the solution has some transition as time pass. As the system starts at rest the initial amplitude is 0, when the forcing is applied to the system the amplitude begins to increase, at this point the forcing term is 'pulling' the system (increasing the amplitude) to its maximum. When the system reaches its maximum amplitude, it becomes in phase with the forcing term. Now as the forcing has lower maximum amplitude than the system it starts to 'pull' the system (this time as a break) to its rest, losing the phase. Finally when the system returns to rest, the process is repeated.

This effect happens because of the boundaries of the system; as the system is bounded reflected waves are present and have an effect over the phase between the system and the forcing and thus affecting the amplitude. The main difference between the damped and undamped system is the fact that after each 'cycle' (from initial to transitional rest position) the amplitude will be diminished in the damped system.

For the solution obtained through Green's function two possibilities are analyzed; a 'near' and a 'far' point:

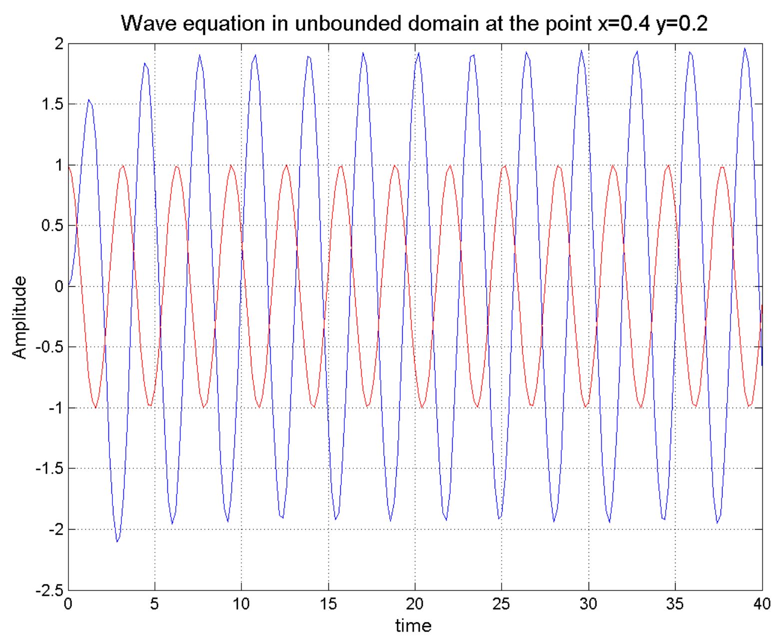

Figure 2-3 Green function, 'near' point

For the point near the wave has the characteristics shown in Figure 2-3 where at start, a transition is present till the system reach its equilibrium. The forcing and the system are almost in opposite phase. What's happening is that the forcing starts at a maximum and generates a positive amplitude in the system, then the forcing decrease and when it reach the zero the system approach its maximum amplitude, then the forcing starts being negative (till its maximum -1) generating a decrease in the amplitude till it reach its minimum. At this point the forcing starts being less negative, but still negative making the system's amplitude to decrease slower. Finally the forcing starts being positive and the process is repeated. In this case there's not a transition between the phase of the system and the forcing. This happens because the system is unbounded and there's no reflection that provokes a change in the initial phase.

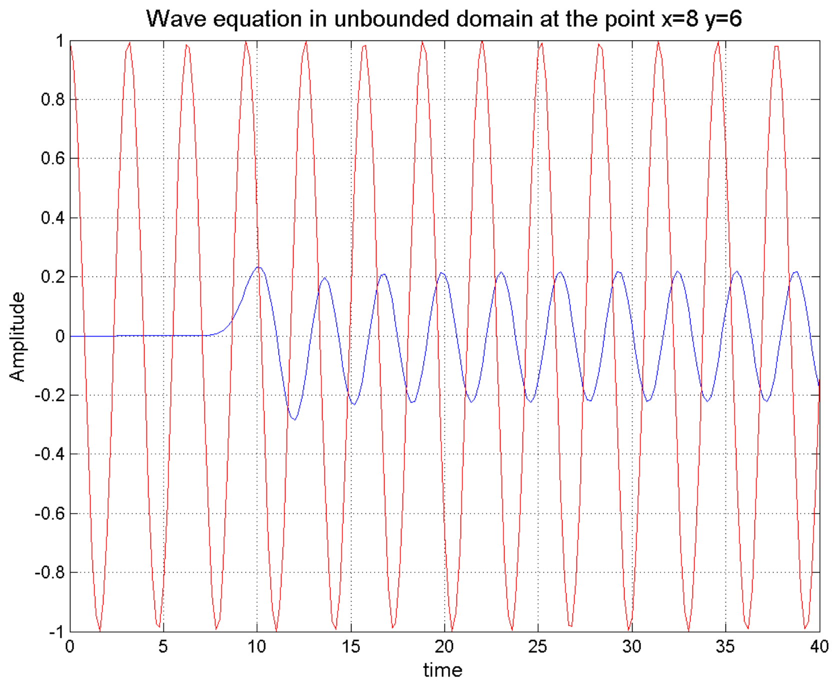

Figure 2-4 Green function, 'far' point

For the 'far' point, the solution is similar in the fact that the forcing and the system aren't in phase. The main difference as expected is the presence of a time delay between the apparition of the forcing and the reaction of the system at this point. Also as expected, the forcing provokes smaller amplitude due to the dissipation caused by the long distance travelled by the wave.

2.2 CPU time

The time used by a CPU to calculate the solution of the system with Green's function is compared with the solution obtained by separation of variables in the same domain Ω in a 100x100 equispaced grid at $t = \pi/2$.

MATLAB is used for this procedure. The results are that the time needed to obtain a snapshot of the solution for the damped and undamped wave equation using spectral representation is quite similar (0.6780 and 0.7055 seconds respectively), which can be considered fast. A different thing happens to elucidate a solution using the Green's function; to have a rough idea the time needed to obtain a solution at 1 point is around 0.2989, which tells us that for a grid of 10 thousand points the time is very superior (around 50 minutes).

2.3 Graphical representations

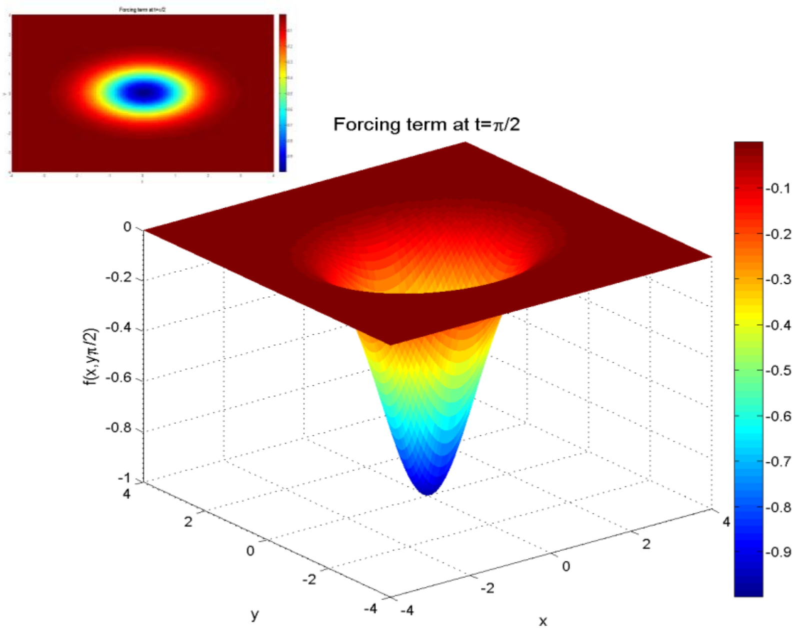

Figure 2-5 Forcing term at $t = \pi/2$

Figure 2-5 represents the forcing term at $t = \pi/2$. It is possible to see that the forcing is concentrated around the origin. In fact, out of a circumference of radius 3 the forcing term is zero. This is shown too in Figure 2-4, because the perturbation takes some time to reach a point far from the origin. The Figure 2-6 is a zoom of the previous graph for showing how the forcing term is in the bounded domain.

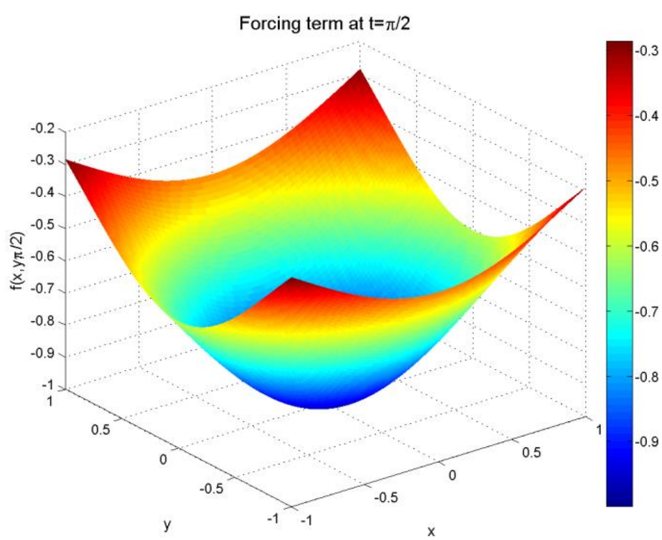

Figure 2-6 Forcing term at $t = \pi/2$ (zoom)

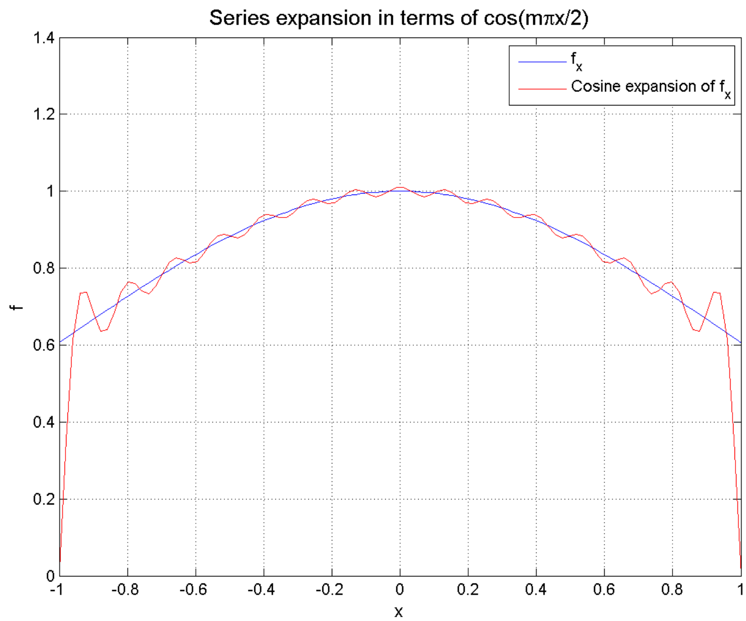

Figure 2-7 Expansion (x)

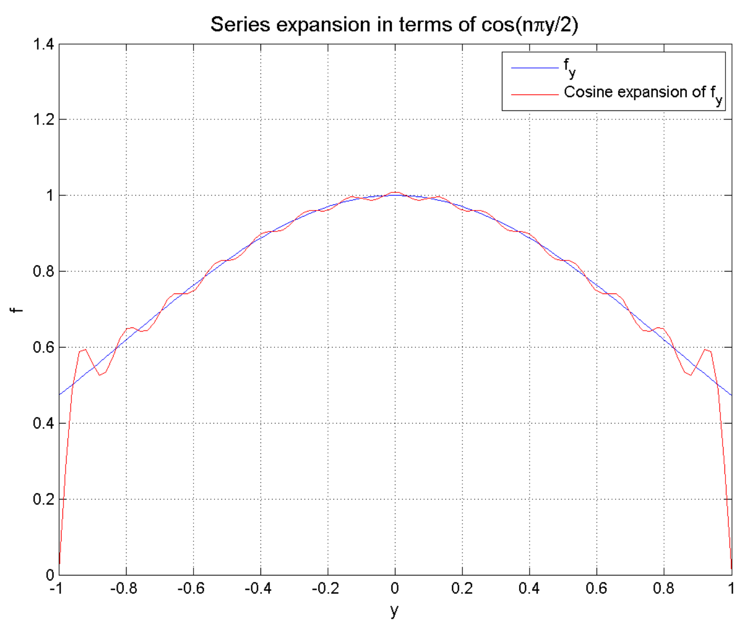

Figure 2-8 Expansion (y)

The expansion in x and y directions are both similar. It is possible to see the Gibbs phenomena because the function $f_x(x)$ and $f_y(y)$ do not verify the boundary conditions. In spite of that, the sequences approximate very well the functions.

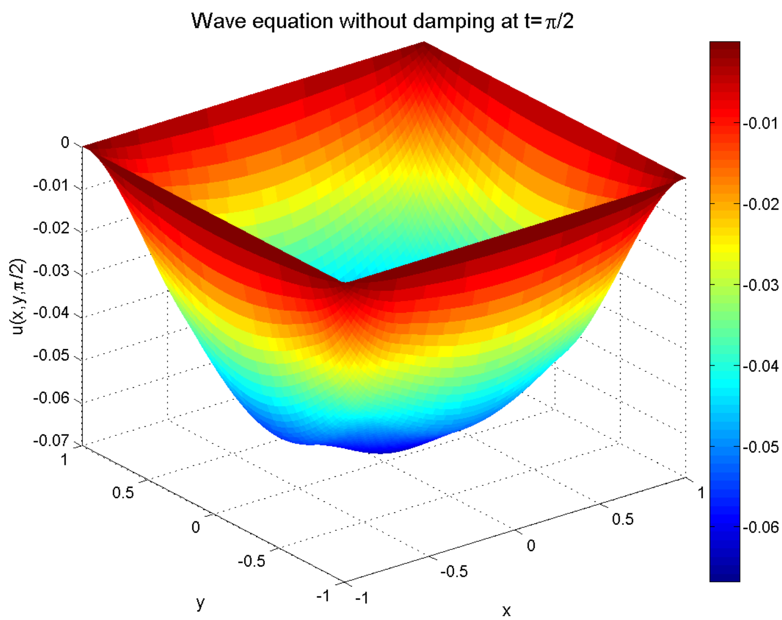

Figure 2-9 System without damping

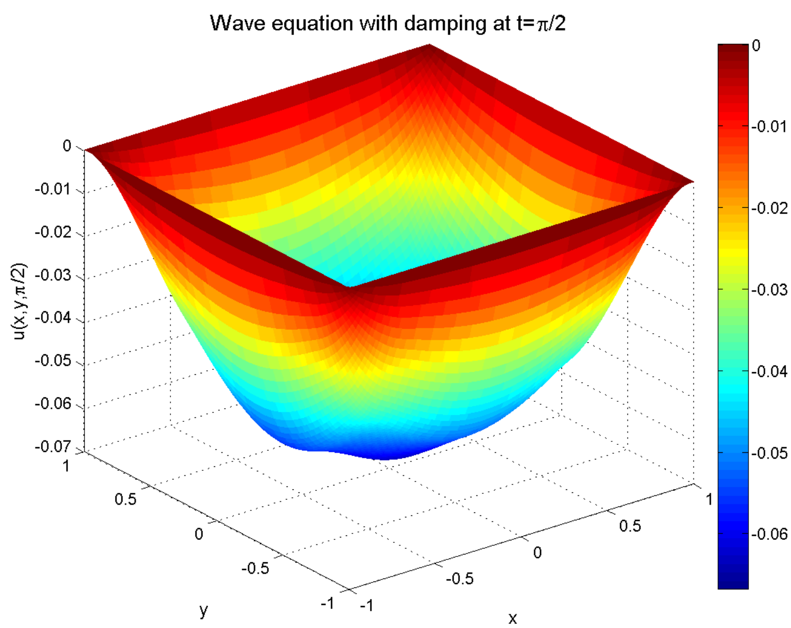

Figure 2-10 System with damping

The two graphs above represent the solution in a bounded domain without damping and with damping respectively at $t = \pi/2$. Both solutions are quite similar, but the damped system presents lower amplitude.